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IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7729–7744 doi:10.1088/1751-8113/40/27/021

The influence of entanglement and decoherence on the quantum Stackelberg duopoly game

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Received 18 January 2007, in final form 10 May 2007
Published 20 June 2007
Online at stacks.iop.org/JPhysA/40/7729

Abstract

In this paper, we investigate the influence of entanglement and decoherence on the quantum Stackelberg duopoly (QSD) game. It is shown that the first-mover advantage can be weakened or enhanced due to the existence of entanglement for the QSD game without decoherence. The influence of decoherence induced by the amplitude damping and the phase damping are explicitly studied in the formalism of Kraus operator representations. We show that the amplitude damping drastically changes the Nash equilibrium of the QSD game and the profits of the two players while the phase damping does not affect the Nash equilibrium and the profits of the two players. It is found that under certain conditions there exists a 'critical point' of the damping parameter for the amplitude damping environment. At the 'critical point' the two players have the same moves and payoffs. The QSD game can change from the first-mover advantage game into the follower-mover advantage game when the damping parameter varies from the left-hand-side regime of the 'critical point' to the right-hand-side regime.

PACS numbers: 03.67.Lx, 03.67.−a, 02.50.Le, 87.23.Ge

1. Introduction

In the past few years, there has been a great deal of effort made to extend the classical game theory into the quantum domain since it is believed that the quantum game theory may be applicable to the study of effective quantum communication as well as for the production of new algorithms for quantum computers. Furthermore, various proposals of applying quantum-like models in social sciences and economics have been put forward [1–7]. In economics, duopoly is a market dominated by two firms large enough to influence the market price. Stackelberg presented a dynamic form of duopoly which is also called the 'leader-follower' model. The

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Stackelberg duopoly game [8–12] is a dynamic extension of the static Cournot duopoly game. Unlike the Cournot duopoly in which both firms make their strategic moves at the same time and thus have to guess what the action of their opponents would be, the Stackelberg duopoly allows one of the firms, say firm A, to move first. Since the other firm, say firm B, can now observe its opponent's move before making its own decision, the game can no longer be modelled as static. In such a dynamic game there exists an interesting result: there is a clear advantage to moving first. This first-mover advantage in the Stackelberg duopoly is due to the fact that being able to make its strategic decision known, firm A does not need to guess what firm B will do because firm B is assumed to behave optimally. Accordingly, firm A can precisely predict firm B's strategic decision and choose its own move in such a way that maximizes its own profits given firm B's choice. This informational advantage is the main driving force behind the first-mover advantage. Recently, the Stackelberg duopoly game has been extended to quantum versions in terms of 'minimal' quantization rules proposed in [13] and the classical-probability action formalism on the initial strategy state [14] for both discrete and continuous variables [15–17]. It has been shown that quantum entanglement affects the first-mover advantage in the classical form. Under certain conditions, the classical situation of the first mover becoming better off and the follower becoming worse off is then avoided in the quantum Stackelberg duopoly (QSD).

As is well known, no system can be completely isolated from its environment. This unwanted system–environment interaction induces entanglement between the quantum system and the environment such that quantum coherence of the system is destroyed and quantum decoherence [18–21] occurs, which results in an inevitable noise in the quantum computation and information processing. Quantum systems are generally very fragile to decoherence which can suppress various nonclassical effects of quantum systems [22–27]. For both players in the QSD game, neither firm A nor firm B can avoid the influence of decoherence induced by their environment. Thus, it is important to analyse effects of quantum decoherence in real practical situations to find out if firm A can continue to maintain its the first-mover advantage in the classical form in a noisy environment. The motivation of this paper is to investigate quantum decoherence how to affect the QSD game in a damping environment and to explore new phenomenon induced by quantum decoherence in the QSD game.

In practice, it is a complicated problem to understand the environment effect on the quantum system. In general, there are three ideal models of noise to describe the environment effect [28], called the amplitude damping, phase damping and the depolarizing channel, respectively. In the present paper, our attention focuses on the amplitude and phase damping since they can capture many of the most important features of the environment noise occurring in quantum-mechanical systems. In order to understand the physical origin of the amplitude and phase damping, let us briefly recall a few basic facts about the interaction between a quantum system and its environment. On one hand, one of the most important reasons for the quantum state change is the energy dissipation of the system induced by the environment. This energy dissipation can be characterized by the amplitude damping model. On the other hand, a state can be a superposition of different states, which is one of the main characteristics of the quantum mechanics. The relative phase and amplitude of the superposed state determines the properties of the whole state. If the relative phases of the superposed states randomly change with the time evolution, then the coherence of the quantum state will be destroyed. This kind of quantum noise process is called the phase damping. In this case, the energy eigenstates of a quantum system do not change as a function of time, but do accumulate a phase which is proportional to the eigenvalue. When the system evolves for an amount of time, partial information about the relative phases between energy eigenstates is lost.

The amplitude and phase damping can be mathematically described by decaying of the diagonal and off-diagonal elements of the reduced density operator of the system. The two effects can be understood in terms of Hamiltonian formalism. If we assume the total Hamiltonian of the system plus environment to be $H_T = H_S + H_R + H_I$, where H_S and H_R are Hamiltonians of the system and environment, respectively, and H_I is the interaction Hamiltonian between them. When the Hamiltonian of the system commutes with that of the interaction between the system and environment, i.e., $[H_S, H_I] = 0$, which means that there is no energy transfer between the system and the environment, energy of the system is conservative, so that what interaction between the system and environment describes is the phase damping effect. When $[H_S, H_I] \neq 0$, there is energy transfer between the system and environment, so that what interaction between the system and environment describes is the amplitude damping effect.

The amplitude and phase damping models have been widely applied to study the influence of the noisy environment on the two-person zero-sum game [29, 30] and the quantum prisoner dilemma [31–35] in terms of the Kraus operator formalism [28, 36, 37]. The purpose of this paper is to study the influence of decoherence induced by the amplitude damping and the phase damping on the QSD game using Kraus operator representation of decoherence [38]. This paper is organized as follows. In section 2, we briefly review the QSD game and investigate the influence of quantum entanglement. In section 3, we analyse the effects of the QSD game under decoherence. We shall conclude our paper with discussions and remarks in the last section.

2. QSD game with entanglement

We consider the QSD game in the classical-probability action formalism with discrete variables [15]. In this formalism, the two players can manipulate the initial strategy state through probabilistically applying two unitary operations. The QSD game can be modelled by two qubits, one for each player of firms A and B who have two possible strategies: identity operator ($\hat{I}$) and the inversion operator ($\hat{C}$). Both $\hat{I}$ and $\hat{C}$ are unitary operators. Assume that each qubit has the basis $|0\rangle$ and $|1\rangle$; then the identity operation $\hat{I}$ preserves consistency in the bases while action of the inversion strategic operation flips the bases, i.e., $\hat{C}|0\rangle = |1\rangle$ and $\hat{C}|1\rangle = |0\rangle$.

Suppose that the QSD game starts with the initial state denoted by a density matrix ρ_i. When two players apply the unitary operators $\hat{I}$ and $\hat{C}$ with probabilities x and $1 - x$ for the first player, y and $1 - y$ for the second player on the two-qubit state, respectively, the initial state ρ_i changes to

$$\rho_f = xyI_A \otimes I_B\rho_i I_A^\dagger \otimes I_B^\dagger + x(1-y)I_A \otimes C_B\rho_i I_A^\dagger \otimes C_B^\dagger \\ + y(1-x)C_A \otimes I_B\rho_i C_A^\dagger \otimes I_B^\dagger + (1-x)(1-y)C_A \otimes C_B\rho_i C_A^\dagger \otimes C_B^\dagger. \quad (1)$$

Assume that in Stackelberg duopoly players' moves are given by probabilities lying in the range (0, 1]. The moves by firms A and B in a classical duopoly game are given by quantities q_1 and q_2 with q_1 and q_2, respectively, being in the range $[0, \infty)$. We assume that firms A and B agree on a function that can uniquely define a real positive number in the range (0, 1] for every quantity q_1 and q_2. A simple such function is $1/(1 + q_i)$, so that firms A and B find the probabilities x and y to be given by

$$x = (1 + q_1)^{-1}, \qquad y = (1 + q_2)^{-1}. \quad (2)$$

The payoffs of the two players in the QSD game are obtained through mean values of their corresponding payoff operators with respect to the final state of the game. The payoff operators of firms A and B can be defined in terms of the projective operators in the two-qubit Hilbert space as

$$\hat{P}_A(q_1,q_2) = \frac{q_1}{q_{12}}[k|00\rangle\langle 00| - |10\rangle\langle 10| - |01\rangle\langle 01|],$$
$$\hat{P}_B(q_1,q_2) = \frac{q_2}{q_{12}}[k|00\rangle\langle 00| - |10\rangle\langle 10| - |01\rangle\langle 01|], \tag{3}$$

where we have introduced the following parameter:

$$q_{12} = [(1+q_1)(1+q_2)]^{-1}. \tag{4}$$

Then, the payoffs of firms A and B are given by the following trace operations:

$$P_A(q_1,q_2) = \mathtt{Tr}[\rho_f \hat{P}_A(q_1,q_2)], \qquad P_B(q_1,q_2) = \mathtt{Tr}[\rho_f \hat{P}_B(q_1,q_2)]. \tag{5}$$

In the QSD game, firm A is a leader and firm B is a follower. Firm A moves first and firm B moves second. The sequence of events is following. (1) Firm A chooses a quantity $q_1 > 0$. (2) Firm B observes q_1 and then chooses a quantity $q_2 \geqslant 0$. (3) Firms A and B apply the operators $\hat{I}$ and $\hat{C}$ such that firms A and B apply $\hat{I}$ with probabilities x and y, respectively. (4) The payoffs to firms A and B are given by equation (5).

We now find the backward-induction outcome in the QSD game. We do it in exactly the same way as is done in the classical game. We consider the situation in which the initial state of the QSD game is a pure state given by

$$|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle, \tag{6}$$

which leads to the following initial density matrix

$$\rho_i = \cos^2\theta|00\rangle\langle 00| + \cos\theta\sin\theta|00\rangle\langle 11| + \cos\theta\sin\theta|11\rangle\langle 00| + \sin^2\theta|11\rangle\langle 11|. \tag{7}$$

Substituting equations (6) and (2) into equation (1), one can obtain the final state of the QSD game given by the following density matrix

$$\rho_f = \rho_{11}|00\rangle\langle 00| + \rho_{14}|00\rangle\langle 11| + \rho_{22}|01\rangle\langle 01| + \rho_{23}|01\rangle\langle 10|$$
$$+ \rho_{32}|10\rangle\langle 01| + \rho_{33}|10\rangle\langle 10| + \rho_{41}|11\rangle\langle 00| + \rho_{44}|11\rangle\langle 11|, \tag{8}$$

where the matrix elements of the density operator are given by

$$\rho_{11} = q_{12}(\cos^2\theta + q_1q_2\sin^2\theta), \qquad \rho_{22} = q_{12}(q_2\cos^2\theta + q_1\sin^2\theta),$$
$$\rho_{33} = q_{12}(q_1\cos^2\theta + q_2\sin^2\theta), \qquad \rho_{44} = q_{12}(q_1q_2\cos^2\theta + \sin^2\theta), \tag{9}$$
$$\rho_{14} = \rho_{41} = q_{12}(1+q_1q_2)\cos\theta\sin\theta, \qquad \rho_{23} = \rho_{32} = q_{12}(q_1+q_2)\cos\theta\sin\theta.$$

The payoffs to firms A and B can be calculated by using equations (3), (5) and (8) with the following expressions:

$$P_A(q_1,q_2) = \tfrac{1}{2}q_1[k(1+q_1q_2) - 2(q_1+q_2) + k(1-q_1q_2)\cos 2\theta],$$
$$P_B(q_1,q_2) = \tfrac{1}{2}q_2[k(1+q_1q_2) - 2(q_1+q_2) + k(1-q_1q_2)\cos 2\theta], \tag{10}$$

which are exactly in the same as the payoffs to firms A and B in the classical Stackelberg model of duopoly [15] when entanglement vanishes i.e., $\theta = 0$.

The backward-induction outcome of the QSD game is found by first finding firm B's reaction to an arbitrary quantity by firm A. Denoting this quantity as $R_2(q_1)$ we find

$$R_2(q_1) = \mathtt{max} P_A(q_1,q_2)$$
$$= \frac{k - 2q_1 + k\cos 2\theta}{4 - 2kq_1 + 2kq_1\cos 2\theta}. \tag{11}$$

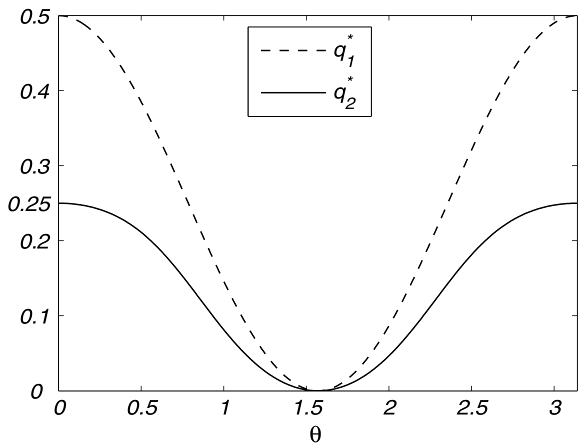

Figure 1. The moves of firms A and B at the subgame perfect Nash equilibrium point for the QSD game without decoherence, q_1^* (the dash line) and q_2^* (the solid line), as functions of the entanglement angle θ when the QSD game is initially in the state given by equation (6) and $k = 1$.

After firm B chooses this quantity $R_2(q_1)$, firm A then finds its optimization problem as

$$\max P_A(q_1, R_2(q_1)) = \tfrac{1}{4}\max[q_1(k - 2q_1 + k\cos 2\theta)], \tag{12}$$

which leads to the backward-induction outcome of the QSD game with the initial state given by equation (6)

$$q_1^* = \frac{1}{2}k\cos^2\theta, \qquad q_2^* = \frac{4k\cos^2\theta}{16 - k^2(1 - \cos 4\theta)}, \tag{13}$$

which is the subgame perfect Nash equilibrium point [12] for the QSD game. Equation (13) indicates that there exists an equilibrium point when $k^2(1 - \cos 4\theta) < 16$ which implies $q_1^* > 0$ and $q_2^* > 0$. At this equilibrium point, using equation (10) we can find payoffs to firms A and B to be given by

$$P_A(q_1^*, q_2^*) = \frac{1}{8}k^2\cos^4\theta, \qquad P_B(q_1^*, q_2^*) = \frac{k^2\cos^4\theta}{16 - k^2(1 - \cos 4\theta)}. \tag{14}$$

From equations (13) and (14), we can see that the existence of quantum entanglement for the initial state of the QSD game affects the equilibrium points of the game and the payoffs to the two players. When the QSD game is initially in an unentangled state $|00\rangle$, i.e., $\theta = 0$, we find the equilibrium point given by $q_1^* = 2q_2^* = k/2$, and payoffs to firms A and B to be $P_A(q_1^*, q_2^*) = 2P_B(q_1^*, q_2^*) = k^2/8$. These results are exactly the same as those in the classical SD game [8, 15]. When the initial state of the game is $|11\rangle$, i.e., $\theta = \pi/2$, we have $q_1^* = q_2^* = 0$ which implies that there does not exist the subgame perfect Nash equilibrium. In order to see the influence of entanglement on the subgame perfect Nash equilibrium and payoffs, we plot the equilibrium point and corresponding profits to the two players with respect to the entanglement angle θ in figures 1 and 2, respectively. From figures 1 and 2, we can see that in the regime of $0 < \theta < \pi/2$ the first-mover advantage is weakened with the increase of the entanglement angle θ, so that entanglement suppresses the first-mover advantage. The follower firm becomes better off and the leader firm becomes worse off. In this regime, entanglement can potentially be a particularly useful element for the 'follower in the leader–follower model. However, the QSD game will show completely different properties

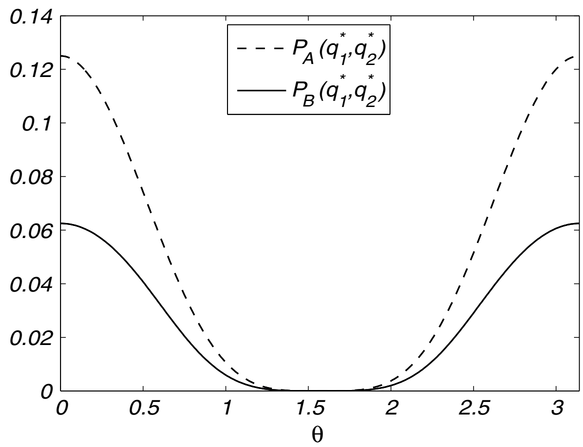

Figure 2. Payoffs to firms A and B at the subgame perfect Nash equilibrium point for the QSD game without decoherence, $P_A(q_1^*, q_2^*)$ (the dash line) and $P_B(q_1^*, q_2^*)$ (the solid line), as functions of the entanglement angle θ when the QSD game is initially in the state given by equation (6) and $k = 1$.

in the entanglement regime of $\pi/2 < \theta < \pi$. In fact, from figures 1 and 2 we can see that in the regime of $\pi/2 < \theta < \pi$ quantum entanglement enhances the first-mover advantage with increase of θ, the leader firm becomes better off and the follower firm becomes worse off, so that entanglement can potentially be a particularly useful element for the 'leader'.

3. QSD game in the noisy environment

In this section, we study the influence of the decoherence on the QSD game. We shall calculate explicitly the subgame perfect Nash equilibrium points of the QSD game in the noisy environment and payoffs to firms A and B at these equilibrium points for the situations of the amplitude damping channel and the phase damping channel, respectively. As is well known, the evolution of quantum states in the noisy environment can be well described in terms of Kraus operators [28, 36, 37]. The quantum system of the initial state ρ_i evolves to the final state ρ_f under the action of the noisy environment. One can relate the initial state and the final state through using a superoperator $\rho_f = \mathcal{S}(\rho_i)$ in which the quantum operation $\mathcal{S}$ on the state ρ_{in} can be described by the Kraus operator sum formalism [1, 2] as

$$\rho_{\text{out}} = \sum_{\mu} M_{\mu}(p)\rho_i M_{\mu}^{\dagger}(p), \tag{15}$$

where p is a parameter to describe the damping produced by the noise environment and it takes its values in the regime of $(0, 1)$. We have $p = 0$ in the absence of the damping while we have $p = 1$ the evolution time approaches the infinity. The two-bit Kraus operators $M_{\mu}(p)$ and $M_{\mu}^{\dagger}(p)$ act on the Hilbert space of the quantum system under consideration, they can be expressed in terms of single-qubit Kraus operators defined below and satisfy the completeness relation $\sum_{\mu} M_{\mu}^{\dagger}(p)M_{\mu}(p) = 1$.

In general, the explicit expressions for the Kraus operators depend on the type of the noisy environment. In what follows we shall consider two kinds of the typical noisy environment, i.e., the amplitude damping and the phase damping. For the single-qubit case, one can use two Kraus operators to describe the amplitude damping. The two Kraus operators are given by

$$m_0(p) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad m_1(p) = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \tag{16}$$

However, in order to describe the phase damping, we have to need three Kraus operators given by

$$m_0(p) = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad m_1(p) = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad m_2(p) = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17}$$

The QSD game between two players proceeds as described in the previous section. However, the two players can now delay their decision to apply the unitary operations $\hat{I}$ and $\hat{C}$ and allow the game to evolve nonunitarily under the noise environment from the initial state of the game ρ_i to the state ρ_i' which acts as the new initial state of the QSD GAME given by

$$\rho_i' = \sum M_\mu(p_1)\rho_i M_\mu^\dagger(p_1), \tag{18}$$

where we have introduced the following two-qubit Kraus operator

$$M_\mu(p_1) = m_r(p_1) \otimes m_s(p_1), \tag{19}$$

where $m_r(p_1)$ are single qubit Kraus operators given by equations (16) and (17) for the amplitude damping and the phase damping, respectively.

After the two players apply the unitary operators $\hat{I}$ and $\hat{C}$ with probabilities x and y on the state ρ_i', respectively, the state ρ_i' evolves into the following state:

$$\begin{aligned}\rho_f = {} & xyI_A \otimes I_B\rho_i' I_A^\dagger \otimes I_B^\dagger + x(1-y)I_A \otimes C_B\rho_i' I_A^\dagger \otimes C_B^\dagger \\ & + y(1-x)C_A \otimes I_B\rho_i' C_A^\dagger \otimes I_B^\dagger + (1-x)(1-y)C_A C_B\rho_i' C_A^\dagger \otimes C_B^\dagger. \end{aligned} \tag{20}$$

If the two players can choose to delay measurements of their payoffs, the game has to experiences the second evolution in the noise environment. The damping induced by the environment can result in the loss of quantum coherence of the state given by equation (20). Under the Kraus operators the state ρ_f becomes

$$\rho_f' = \sum M_\mu(p_2)\rho_f M_\mu^\dagger(p_2), \tag{21}$$

where $M_\mu(p_2) = m_r(p_2) \otimes m_s(p_2)$. Equation (21) can finally lead to the payoffs to the two players through the trace operations given by equation (5). In what follows we shall investigate the influence of the amplitude-damping and phase-damping environment on the QSD game, respectively.

3.1. The amplitude damping channel

We consider the situation in which the two decohering processes described in equations (18) and (21) are the amplitude damping with damping parameters being p_1 and p_2, respectively. By substituting the Kruas operators to describe the amplitude damping into equations (18) and (21), we arrive at the final backward-induction outcome

$$\begin{aligned}\rho_f' = {} & \rho_{11}'|11\rangle\langle 11| + \rho_{14}'|11\rangle\langle 22| + \rho_{22}'|12\rangle\langle 12| + \rho_{23}'|12\rangle\langle 21| \\ & + \rho_{32}'|21\rangle\langle 12| + \rho_{33}'|21\rangle\langle 21| + \rho_{41}'|22\rangle\langle 11| + \rho_{44}'|22\rangle\langle 22|, \end{aligned} \tag{22}$$

where the matrix elements of the density operator are given by

$$
\begin{aligned}
\rho'_{11} &= q_{12}\big\{\cos^2\theta + q_1q_2\sin^2\theta + (q_1+q_2)p_2 + (1-q_1)(1-q_2)p_1^2(1-p_2)^2\sin^2\theta \\
&\quad + (\sin^2\theta + q_1q_2\cos^2\theta)p_2^2 + p_1(1-p_2) \\
&\quad \times [q_2 + q_1(1-2q_2) + (2-q_1-q_2)p_2]\sin^2\theta\big\}, \\
\rho'_{14} &= \rho'_{41} = \tfrac{1}{2}q_{12}(1-p_1)(1-p_2)(1+q_1q_2)\sin(2\theta), \\
\rho'_{22} &= q_{12}(1-p_2)\{(1+q_1p_2)q_2\cos^2\theta + [1-(1-q_2)p_1] \\
&\quad \times [q_1 + p_2 + (1-q_1)(1-p_2)p_1]\sin^2\theta\}, \\
\rho'_{23} &= \rho'_{32} = \tfrac{1}{2}q_{12}(1-p_1)(1-p_2)(q_1+q_2)\sin(2\theta), \\
\rho'_{33} &= q_{12}(1-p_2)\{(1+q_2p_2)q_2\cos^2\theta + [1-(1-q_1)p_1] \\
&\quad \times [q_1 + p_2 + (1-q_2)(1-p_2)p_1]\sin^2\theta\}, \\
\rho'_{44} &= (1-p_2)^2[q_1q_2\cos^2\theta + (1-(1-q_1)p_1)(1-(1-q_2)p_1)\sin^2\theta,
\end{aligned} \tag{23}
$$

which indicate that the amplitude damping not only affects the non-diagonal elements of the density operator of the two-qubit system under our consideration but also changes the diagonal elements of the density operator of the system. In what follows, we will show that it is the change of the diagonal elements that leads to the amplitude damping drastically changing the Nash equilibria of the game.

Substituting equation (22) into equation (5), we can obtain payoffs of firms A and B

$$
\begin{aligned}
P_A(q_1,q_2) &= \frac{q_1}{q_{12}}(k\rho'_{11} - \rho'_{22} - \rho'_{33}), \\
P_B(q_1,q_2) &= \frac{q_2}{q_{12}}(k\rho'_{11} - \rho'_{22} - \rho'_{33}).
\end{aligned} \tag{24}
$$

Following the method in the previous section, we can obtain the subgame perfect Nash equilibrium point for the QSD game with the amplitude damping

$$
\begin{aligned}
q_1^* &= \frac{k\cos^2\theta + A_1(p_1,p_2)}{2 + B_1(p_1,p_2)}, \\
q_2^* &= \frac{k\cos^2\theta + A_2(p_1,p_2)}{[16 - k^2(1-\cos 4\theta)]/4 + B_2(p_1,p_2)},
\end{aligned} \tag{25}
$$

where we have introduced the damping functions

$$
\begin{aligned}
A_1(p_1,p_2) &= \{-2(p_1+p_2-p_1p_2) + (2+k)p_2[p_2 + (1-p_1)(2p_1+p_2-p_1p_2)]\}\sin^2\theta, \\
B_1(p_1,p_2) &= -2[(1+k)p_2 + (2+k)p_1(1-p_1)(1-p_2)^2\sin^2\theta], \\
A_2(p_1,p_2) &= A_1 + \tfrac{1}{2}(k\cos^2\theta + A_1)B_1, \\
B_2(p_1,p_2) &= 2(2+k)^2p_1^3(-2+p_1)(1-p_2)^4\sin^4\theta + 2p_1(1-p_2)^2\sin^2\theta\big\{-8 - 2k + k^2 \\
&\quad + 2(2+5k+2k^2)p_2 + k(k+4p_2+2kp_2)\times\cos 2\theta + 2(4+4k+k^2)p_2^2\cos^2\theta \\
&\quad - p_1[-10-4k+(8+12k+4k^2)p_2] + (2+2k+k^2-2(2+k)p_2 \\
&\quad + \big(2+2k+k^2-2(2+k)p_2 + (4+4k+k^2)p_2^2\big)\cos 2\theta\big\} \\
&\quad + p_2\big\{-8-5k+\big(3+5k+\tfrac{5}{2}k^2\big)p_2 + (2+k)p_2^2 - \big(1+k+\tfrac{1}{4}k^2\big)p_2^3 \\
&\quad + \big[k + (1-k-\tfrac{1}{2}k^2)p_2 - (2+k)p_2^2 + \big(1+k+\tfrac{1}{4}k^2\big)p_2^3\big]\cos 4\theta\big\},
\end{aligned} \tag{26}
$$

At above equilibrium point, payoffs of firms A and B are given by

$$P_A(q_1^*, q_2^*) = \frac{[k\cos^2\theta + C_1(p_1, p_2)]^2}{8 + D_1(p_1, p_2)},$$
$$P_B(q_1^*, q_2^*) = \frac{[1 + C_2(p_1, p_2)][k\cos^2\theta + C_3(p_1, p_2)]^2}{16 - k^2(1 - \cos 4\theta) + D_2(p_1, p_2)}, \tag{27}$$

where the damping functions are given by

$$C_1(p_1, p_2) = C_3(p_1, p_2) = A_1(p_1, p_2),$$
$$D_1(p_1, p_2) = 8C_2(p_1, p_2) = 4B_1(p_1, p_2), \tag{28}$$
$$D_2(p_1, p_2) = 4B_2(p_1, p_2).$$

From equations (25) and (27) we can see that the without-damping solutions given by equations (13) and (14) are recovered easily since we have $C_i(0, 0) = D_i(0, 0) = 0$ when the damping vanishes, i.e., $p_1 = p_2 = 0$.

In order to observe the influence of the amplitude damping, we consider the situation in which the game experiences only the second decohering process while the first decohering process vanishes, i.e., $p_1 = 0$ and $p_2 \neq 0$. In this case, the related damping functions in equation (26) are given by

$$A_1(0, p_2) = -4p_2,$$
$$A_2(0, p_2) = -p_2[2 - p_2(5 - 3p_2)\sin^2\theta], \tag{29}$$
$$B_2(0, p_2) = -p_2\left[\left(52 - 42p_2 - 3p_2^2\right) - \left(4 - 2p_2 - 6p_2^2 + 9p_2^3\right)\cos(4\theta)\right]/4,$$

where we have taken $k = 1$.

In particular, when $\theta = 0$, namely, the initial entanglement vanishes, we obtain

$$q_1^* = \frac{1}{2 - 4p_2}, \qquad q_2^* = \frac{1 - 2p_2}{4 - 12p_2 + 10p_2^2},$$
$$P_A(q_1^*, q_2^*) = \frac{1}{8 - 16p_2}, \tag{30}$$
$$P_B(q_1^*, q_2^*) = \frac{1 - 2p_2}{8\left(2 - 6p_2 + 5p_2^2\right)},$$

which implies that there exists the subgame perfect Nash equilibrium point for the QSD game with the amplitude damping in the damping regime of $p_2 < 1/2$ since in this damping regime $q_1^* > 0, q_2^* > 0$, and $P_{A,B}(q_1^*, q_2^*) > 0$. In figures 3 and 4, we have plotted the subgame perfect Nash equilibrium point and corresponding payoffs to the two players with respect to the damping parameter p_2, respectively. From figures 3 and 4, we can see that when the QSD game is initially in the unentangled state, the amplitude damping can enhance the first-mover advantage with increasing the damping parameter p_2.

However, the situation will be different when the QSD game is initially in an entangled state. For instance, when the QSD game is initially in the maximally entangled state, i.e., $\theta = \pi/4$, we have

$$q_1^* = \frac{1 - p_2(2 - 3p_2)}{4 - 8p_2},$$
$$q_2^* = \frac{1 - [4 - p_2(7 - 6p_2)]p_2}{7 - p_2\{28 - p_2[22 + 3p_2(4 - 3p_2)]\}},$$
$$P_B(q_1^*, q_2^*) = \frac{(1 - 2p_2)[1 - p_2(2 - 3p_2)]^2}{8\{[7 - p_2[28 - p_2(22 + 3p_2(4 - 3p_2))]\}},$$

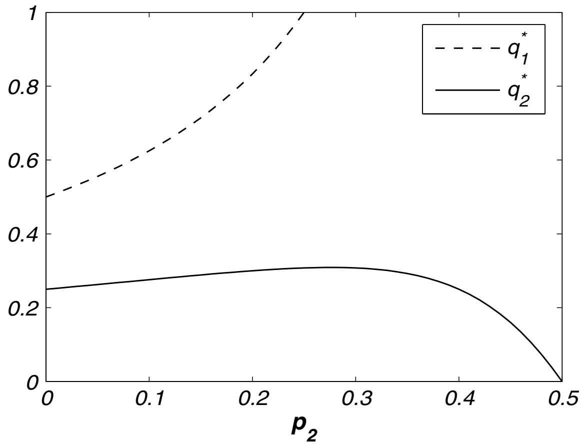

Figure 3. The moves of firms A and B at the subgame perfect Nash equilibrium point for the QSD game with only the second amplitude damping process described by the parameter p_2, q_1^* (the dash line) and q_2^* (the solid line), as functions of the damping parameter p_2 when the QSD game is initially in the state given by equation (6) with $\theta = 0$ and $k = 1$.

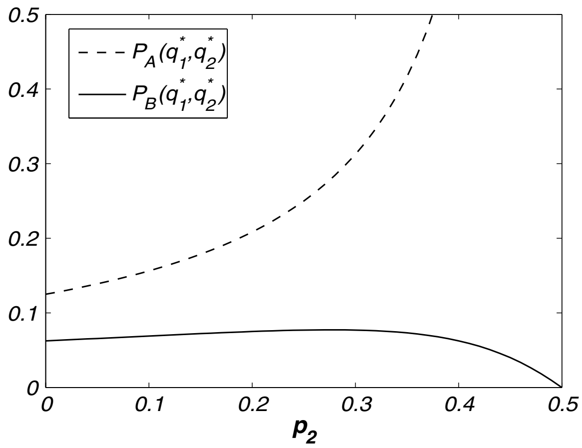

Figure 4. Payoffs to firms A and B at the subgame perfect Nash equilibrium point for the QSD game with only the second amplitude damping process described by the parameter p_2, $P_A(q_1^*, q_2^*)$ (the dash line) and $P_B(q_1^*, q_2^*)$ (the solid line), as functions of the damping parameter p_2 when the QSD game is initially in the state given by equation (6) with $\theta = 0$ and $k = 1$.

$$P_A(q_1^*, q_2^*) = \frac{[1 - p_2(2 - 3p_2)]^2}{32 - 64p_2}. \tag{31}$$

In figures 5 and 6, we have plotted the subgame perfect Nash equilibrium point and corresponding payoffs to the two players with respect to the damping parameter p_2, respectively. From figures 5 and 6, we can see that there is a ‘critical point’ labelled by C for q_1^* and q_2^*, or $P_A(q_1^*, q_2^*)$ and $P_B(q_1^*, q_2^*)$. At the ‘critical point’, the two players have the same payoffs, i.e., $q_1^* = q_2^*$ and $P_A(q_1^*, q_2^*) = P_B(q_1^*, q_2^*)$. This implies that the first-mover advantage completely disappears due to the influence of the amplitude damping of the noise

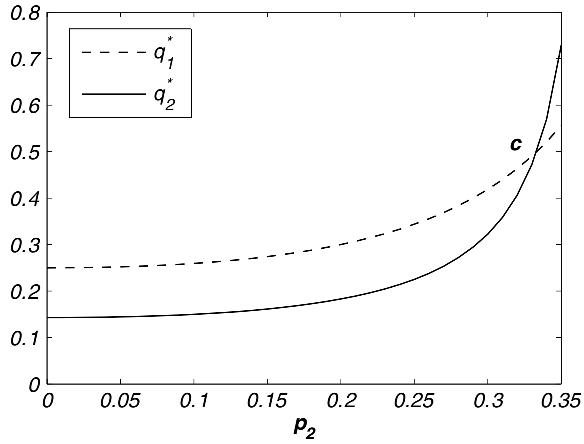

Figure 5. The moves of firms A and B at the subgame perfect Nash equilibrium point for the QSD game with only the second amplitude damping process described by the parameter p_2, q_1^* (the dash line) and q_2^* (the solid line), as functions of the damping parameter p_2 when the QSD game is initially in the state given by equation (6) with $\theta = \pi/4$ and $k = 1$.

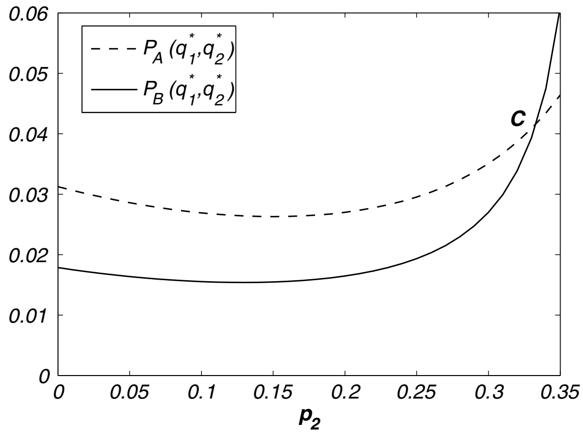

Figure 6. Payoffs to firms A and B at the subgame perfect Nash equilibrium point for the QSD game with only the second amplitude damping process described by the parameter p_2, $P_A(q_1^*, q_2^*)$ (the dash line) and $P_B(q_1^*, q_2^*)$ (the solid line), as functions of the damping parameter p_2 when the QSD game is initially in the state given by equation (6) with $\theta = \pi/4$ and $k = 1$.

environment. From equations (25)–(27) we find the 'critical point' to be

$$\begin{aligned} &p_2 = \tfrac{1}{3}, \qquad q_1^* = q_2^* = \tfrac{1}{2}, \\ &P_A(q_1^*, q_2^*) = P_B(q_1^*, q_2^*) = \tfrac{1}{24}. \end{aligned} \tag{32}$$

Figures 5 and 6 indicate that the influence of the damping on the subgame perfect Nash equilibrium point and corresponding payoffs to the two players is different within different damping regimes. It is interesting to note that the QSD game can change from the first-mover advantage game into the follower-mover advantage game when the damping parameter p_2 varies from the left-hand-side damping regime of the 'critical point' to the right-hand-side damping regime. On the left-hand side of the 'critical point', the first-mover advantage is

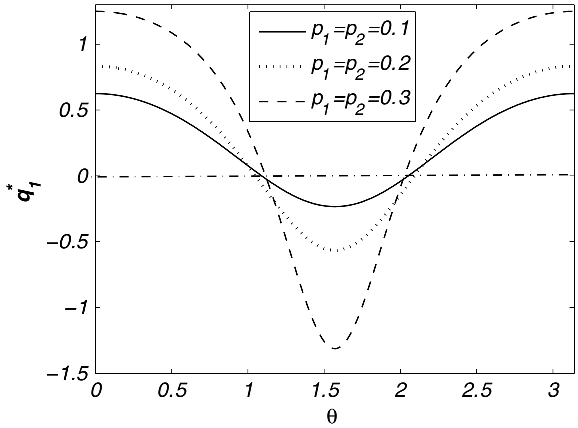

Figure 7. The move of firm A at the subgame perfect Nash equilibrium point for the QSD game with two amplitude damping processes described by the parameters p_1 and p_2, q_1^*, as the function of the entanglement angle θ when the QSD game is initially in the state given by equation (6) and $k = 1$. The solid, star and dash lines correspond to the cases of $p_1 = p_2 = 0.1, 0.2$, and 0.3, respectively.

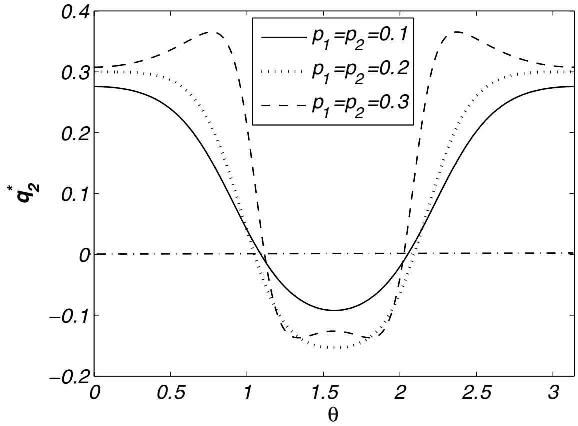

Figure 8. The move of firm B at the subgame perfect Nash equilibrium point for the QSD game with two amplitude damping processes described by the parameters p_1 and p_2, q_2^*, as the function of the entanglement angle θ when the QSD game is initially in the state given by equation (6) and $k = 1$. The solid, star and dash lines correspond to the cases of $p_1 = p_2 = 0.1, 0.2$, and 0.3, respectively.

weakened with the increase of the damping parameter p_2 while on the right-hand side of the 'critical point' the follower-mover advantage is enhanced with the increase of the damping parameter p_2.

When two decohering processes described in equations (18) and (21) are taken account into, the situation will become more complicated. In order to see the influence of the two damping parameters p_1 and p_2 on the subgame perfect Nash equilibrium point for an arbitrary value of the entangling parameter θ, using the expressions given by equation (25) we have plotted q_1^* and q_2^* with respect to variation of θ in figures 7 and 8, respectively. Here we have taken $p_1 = p_2$ and $k = 1$. From figures 7 and 8 we can see that for each pair of damping parameters (p_1, p_2), there are two zero-points which can be labelled by θ_1 and θ_2 with $\theta_1 < \theta_2$,

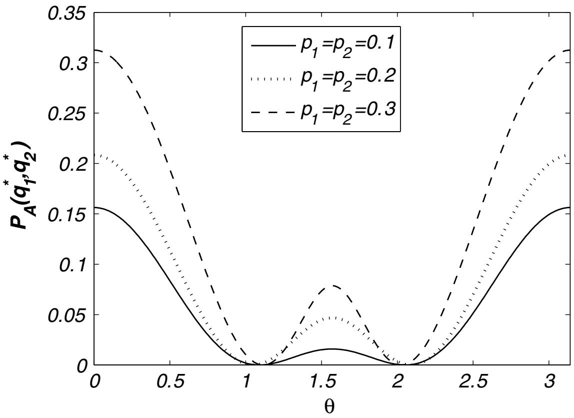

Figure 9. Payoff to firm A at the subgame perfect Nash equilibrium point for the QSD game with two amplitude damping processes described by the parameters p_1 and p_2, $P_A(q_1^*, q_2^*)$, as the function of the entanglement angle θ when the QSD game is initially in the state given by equation (6) and $k = 1$. The solid, star and dash lines correspond to the cases of $p_1 = p_2 = 0.1, 0.2$ and 0.3, respectively.

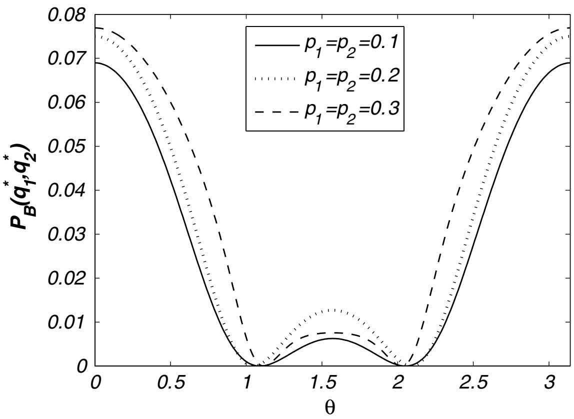

Figure 10. Payoff to firm B at the subgame perfect Nash equilibrium point for the QSD game with two amplitude damping processes described by the parameters p_1 and p_2, $P_B(q_1^*, q_2^*)$, as the function of the entanglement angle θ when the QSD game is initially in the state given by equation (6) and $k = 1$. The solid, star and dash lines correspond to the cases of $p_1 = p_2 = 0.1, 0.2$ and 0.3, respectively.

respectively. When $\theta_1 < \theta < \theta_2$, both q_1^* and q_2^* are negative (corresponding curves have been cut out in the figures). Therefore, there does not exist the subgame perfect Nash equilibrium point in this regime since the existence of the equilibrium point requires both q_1^* and q_2^* to be positive. Using the expressions given by equation (27), we have also plotted the corresponding payoffs to the two players in figures 9 and 10 which indicate that payoffs to both of two players can be improved with the increase of the damping parameters.

3.2. The phase damping channel

We now turn to the situation of the phase damping channel. In this case, each decohering period can be described in terms of three Kraus operators given by equation (17). After two times of phase decohering, the final state of the QSD game given by equation (21) becomes

$$\rho'_f = \rho'_{11}|11\rangle\langle 11| + \rho'_{14}|11\rangle\langle 22| + \rho'_{22}|12\rangle\langle 12| + \rho'_{23}|12\rangle\langle 21| + \rho'_{32}|21\rangle\langle 12| + \rho'_{33}|21\rangle\langle 21| + \rho'_{41}|22\rangle\langle 11| + \rho'_{44}|22\rangle\langle 22|, \quad (33)$$

where the matrix elements of the density operator are given by

$$\begin{aligned}
\rho'_{11} &= q_{12}(\cos^2\theta + q_1 q_2 \sin^2\theta),\\
\rho'_{22} &= q_{12}(q_2\cos^2\theta + q_1 \sin^2\theta),\\
\rho'_{33} &= q_{12}(q_1\cos^2\theta + q_2 \sin^2\theta),\\
\rho'_{44} &= q_{12}(q_1 q_2\cos^2\theta + \sin^2\theta),\\
\rho'_{14} &= \rho'_{41} = \tfrac{1}{2}q_{12}(1 + q_1 q_2)(1 - p_1)^2(1 - p_2)^2\sin(2\theta),\\
\rho'_{23} &= \rho'_{32} = \tfrac{1}{2}q_{12}(1 - p_1)^2(1 - p_2)^2\sin(2\theta).
\end{aligned} \quad (34)$$

Comparing equations (33) and (34) with equations (8) and (9), we can see that the phase damping affects only the off-diagonal elements of the output state ρ'_f while the diagonal elements of the output state ρ'_f remain unchanged. It is this point that reveals characteristics of the phase damping. On the other hand, from definition of the payoffs for firms A and B given by equations (3) and (4) we can observe the fact that only the diagonal elements of the output state of the game contribute to their payoffs. Therefore, we can conclude that the phase damping does not affect the payoffs to the two players.

4. Conclusions

In conclusion, we have studied the influence of entanglement and decoherence on the discrete-variable QSD game in the Kraus-operator formalism. We have observed novel features in the QSD game of interest, which are completely due to quantum entanglement and decoherence. It has been shown that the QDS game can exhibit rich subgame-perfect-equilibria or backward-induction structures due to the presence of entanglement and decoherence. We have shown that in different entangling regimes the first-mover advantage can be weakened or enhanced due to the existence of the initial quantum entanglement for the QSD game without decoherence. We have calculated the subgame perfect Nash equilibrium of the QSD game in the presence of both entanglement and decoherence in terms of Kraus-operator representations. The effects of the amplitude damping and the phase damping on the subgame perfect Nash equilibrium of the QSD game and the payoffs to the two players at the equilibrium points are explicitly studied. It has been shown that the amplitude damping seriously affects the backward-induction outcome of the QSD game while the phase damping does not affect both of the subgame perfect Nash equilibrium and the profits of the two players. Physically, this is because in the present formalism of game quantization, the payoff operators of players are actually 'mixed states' denoted by 'density operators' involving only *incoherent* superpositions in a two-qubit Hilbert space. Themselves of these payoff operators do not exhibit any quantum coherence, so their meanvalues with respect to an arbitrary state in the two-qubit Hilbert space depends on only the diagonal elements of the density operator of denoting the state. What the amplitude damping describes is the decay of both diagonal and non-diagonal elements of the density operator of the system. Hence, the amplitude damping can drastically change the Nash equilibrium and the payoffs of the two players. However, the situation is different for the phase damping. what the phase damping describes is just the loss of quantum coherence, i.e., the decay of the non-diagonal elements of the density operator of the system. Therefore, the phase damping cannot affect the Nash equilibrium and the payoffs of the two players. This reveals new insight of quantum coherence in quantum games.

It is worth mentioning that when the QSD game is initially in the maximally entangled state, we have found a 'critical point' for the moves and payoffs to the two players in the presence of the amplitude damping. The appearance of the 'critical point' is a new phenomenon induced by the amplitude damping. At the 'critical point' the two players have the same moves and payoffs. The QSD game can change from the first-mover advantage game into the follower–mover advantage game when the damping parameter varies from the left-hand-side damping regime of the 'critical point' to the right-hand-side damping regime with the time evolution or the increase of the damping. And on the two sides of the 'critical point' the QSD game with the amplitude damping can exhibit completely different characteristics for the backward-induction outcome. On the left-hand side of the 'critical point' the first-mover advantage is weakened with the increase of the damping parameter while on the right-hand side of the 'critical point' the follower–mover advantage is enhanced with the increase of the damping parameter. Finally, we hope that these new features revealed in the present paper for the QSD game with entanglement and decoherence would be of interest not only to the quantum information field but also to the applied economics.

Acknowledgments

This work was supported the National Fundamental Research Program of China grant no. 2001CB309310, the National Natural Science Foundation of grant no. 10325523, and the Education Department of Hunan Province.

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